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Prolongation Structure of a New Integrable System

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We have obtained the inverse scattering equations associated with a new pair of coupled nonlinear evolution equations in two dimensions. The spectral parameter is introduced by invoking the invariance of the equation set, and imposing those on the Lax pair.

Completely integrable nonlinear partial differential equations are of wide interest owing to the underlying rich structure of the solution manifold (Rebbi and Soliani, 1984) and infinite number of generalized Lie symmetries (Anderson and Ibragimov, 1979). Here we report a pair of new coupled partial differential equations in two dimensions which is shown to possess a Lax pair (Lax, 1968). Our approach is that of Whalquist and Eastabrook (1975) based on the idea of prolongation structure. The equation set we propose to study is

$$\begin{gathered} \rho u_{3x} - K\rho_x u_{2x} + 6uu_x\rho + \rho u_t = 0 \\ \rho_t + 2(\rho u_x + u\rho_x) = 0 \end{gathered} \tag{1}$$

where K is an arbitrary constant. We proceed by defining the set of independent variables

$$\begin{gathered} u_x = p \\ u_{xx} = q \end{gathered} \tag{2}$$

Then (1) and (2) are seen to be proper sections of the following differential two-forms:

$$\begin{aligned} \Delta_1 &= du \wedge dt - p\, dx \wedge dt \\ \Delta_2 &= dp \wedge dt - q\, dx \wedge dt \\ \Delta_3 &= \rho\, dq \wedge dt - \rho\, du \wedge dx + 6\rho up\, dx \wedge dt - Kq\, d\rho \wedge dt \\ \Delta_4 &= -dp \wedge dx + 2p\rho\, dx \wedge dt + 2u\, d\rho \wedge dt \end{aligned} \tag{3}$$

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It is then easy to observe that these systems of differential forms are closed under exterior differentiation, that is,

$$d\Delta_i = \sum \beta_{iK}\Delta_K \tag{4}$$

β_{iK} are some functions of x and t.

In the prolongation approach of Whalquist and Eastabrook it is then required to search for a one-form (Choudhury and Roy, 1979, 1980)

$$\omega_i = dy_i + F_i\,dx + G_i\,dt \tag{5}$$

with $F_i = F_i(x, t, \rho, u, p, q, y_i)$ and $G_i = G_i(x, t, \rho, u, p, q, y_i)$ such that the exterior derivative of ω_i is in the ideal generated by the forms Δ, and ω_i, that is,

$$d\omega n = \sum \Delta_j f_{nj} + \sum (a_i^n\,dx + b_i^n\,dt)\wedge\omega_i \tag{6}$$

Substituting the explicit forms of the Δ_i's and ω_i's in equation (6) we obtain

$$\begin{aligned}
&\sum (F_\mu^m\,d\mu\wedge dx + G_\mu^m\,d\mu\wedge dt) - f_{n1}(du\wedge dt - p\,dx\wedge dt)\\
&\quad -f_{n2}(dp\wedge dt - q\,dx\wedge dt) - f_{n3}\\
&\quad \times(dq\wedge dt - du\wedge dx + [6up - q\gamma]\,dx\wedge dt)\\
&\quad -f_{n4}(dx\wedge dt - \gamma\,dx\wedge dt) - f_{n5}(-d\lambda\wedge dx + [2u\gamma + 2Kp]\,dx\wedge dt)\\
&\quad -(a_i^n\,dx + b_i^n\,dt)\times(F_i\,dx + G_i dt + dy_i) = 0
\end{aligned} \tag{7}$$

Equating coefficients of several two-forms, we obtain the following equations, yielding information about the structures of F and G:

$$\begin{gathered}
F_u = -G_q, \qquad KqF_u - 2uF_\rho\rho = \rho G_\rho\\
pG_u + qG_p + 6upF_u + 2p\rho F_\rho = [F, G]
\end{gathered} \tag{8}$$

By repeated differentiation we have

$$\begin{aligned}
&G_{qu} = G_{pq} = G_{pp} = 0\\
&G_{uuu\rho} = 0\\
&G_{qq} = 0 = G_{pp}\\
&G_{uuu} = 0\\
&\rho G_{\rho q} = KF_u
\end{aligned} \tag{9}$$

From these equations we can construct the most general form of F and G which will yield the required matrix structure through the nonlinear term in equation (8), and this can actually be done for any value of K, and the forms of F and G explicitly depend on K. But for simplicity we consider only a few simple values for K and show how the method works.

Case I ($K=-1$). In this case by solving the above equations we observe that F and G can be written as

$$\begin{aligned} F &= -\frac{1}{\rho}x_1 - \rho u x_2 \\ G &= p x_3 + \frac{2u}{\rho}\cdot x_1 + 2u^2\rho x_2 + q\rho x_2 \end{aligned} \tag{10}$$

along with

$$\begin{aligned} [x_1, x_2] &= -x_3 \\ [x_1, x_3] &= -4x_1 \\ [x_2, x_3] &= 4x_2 \end{aligned} \tag{11}$$

so we can write down the Lax pair as

$$\begin{aligned} y_x &= F_y \\ y_t &= Gy \end{aligned} \tag{12}$$

where y is a three-component vector $y \equiv (y_1, y_2, y_3)$ and

$$F = \begin{pmatrix} 0 & 0 & 4/p \\ 0 & 0 & -4\rho u \\ -u\rho 1/\rho 0 \end{pmatrix} \tag{13}$$

$$G = \begin{pmatrix} 4p & 0 & -8u/\rho \\ 0 & -4p & 8u^2\rho + 4\rho q \\ 2u^2\rho + \rho q & -2u/\rho & 0 \end{pmatrix} \tag{14}$$

Case II ($K=+1$). In this situation F and G are written as

$$\begin{aligned} F &= \rho\bar{x}_1 + \frac{u}{\rho}\bar{x}_2 \\ G &= u_x\bar{x}_3 - 2u\rho\bar{x}_1 - (2u^2 + u_{xx})\frac{1}{\rho}\bar{x}_2 \end{aligned}$$

along with the commutation rules

$$\begin{aligned} [\bar{x}_1, \bar{x}_2] &= -\bar{x}_3 \\ [\bar{x}_1, \bar{x}_3] &= 0 \\ [\bar{x}_2, \bar{x}_3] &= 0 \end{aligned} \tag{16}$$

Case III ($K=2$). For this value of K the forms of F and G are given as

$$F=\frac{\rho^2}{4}\tilde{x}_1+\frac{4u}{\rho^2}\tilde{x}_2$$
$$G=u_x\tilde{x}_3-\frac{u\rho^2}{2}\tilde{x}_1-\frac{1}{\rho^2}(8u^2+4u_{xx})\tilde{x}_2 \tag{17}$$

The Lie algebra of $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ is governed by

$$[\tilde{x}_1,\tilde{x}_3]=2\tilde{x}_1$$
$$[\tilde{x}_2,\tilde{x}_3]=-2\tilde{x}_2 \tag{18}$$
$$[\tilde{x}_1,\tilde{x}_2]=-\tilde{x}_3$$

so we can again write down the explicit matrix forms of the Lax problem as

$$F=\begin{pmatrix}0 & 0 & \rho2/2\\ 0 & 0 & -8u/\rho^2\\ 4\mu/\rho^2 & -\rho^2/4 & 0\end{pmatrix} \tag{19}$$

$$G=\begin{pmatrix}-2p & 0 & -u\rho^2\\ 0 & 2p & (8/\rho^2)(2u^2+u_{xx})\\ -(4/\rho^2)(2u^2+u_{xx}) & u\rho^2/2 & 0\end{pmatrix} \tag{20}$$

Introduction of the Spectral Parameter. The implementation of the famous inverse scattering transform is possible only if the Lax pair contains an eigenvaluelike parameter. But none of the equations deduced above contain any spectral parameters. To introduce such a parameter we invoke the symmetry of the nonlinear equation itself and impose it on the linear set. Let us consider the case $K=-1$, where we can deduce that the nonlinear equations are invariant under the following transformations:

$$x\to\mu^{1/3}x;\qquad t\to\mu t;\qquad u\to\mu^{-2/3}u,\qquad \rho\to\rho \tag{21}$$

Then the Lax pair becomes

$$y_x=F'y,\qquad y_t=G'y \tag{22}$$

with

$$F'=\begin{pmatrix}0 & 0 & (4/\rho)\mu^{1/3}\\ 0 & 0 & -4\rho u\mu^{-1/3}\\ -\rho u\mu^{-1/3} & \rho^{-1}\mu^{1/3} & 0\end{pmatrix} \tag{23}$$

$$G'=\begin{pmatrix}4u_x & 0 & (8u/\rho)\mu^{1/3}\\ 0 & -4u_x & (8u^2p+4\rho u_{xx})\mu^{-1/3}\\ (2u^2\rho+\rho u_{xx})\mu^{-1/3} & -(2u/\rho)u^{-1/3} & 0\end{pmatrix} \tag{24}$$

Similar calculations can also be done for other values of K.

In the above paragraphs we have presented a new integrable nonlinear coupled partial differential equation from the point of view of prolongation structure. We have determined the pseudopotential structure along with the underlying Lie algebra. Lastly the problem of introducing spectral parameters has been solved for the case with $K=-1$. Discussions about the other properties (such as infinite set of conserved quantities, the implementation of IST itself, and existence of extended Lie symmetries) will be communicated in a future publication.

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